Episode 3

Motion of Particles:
Normal-Tangential and Polar Coordinates

ENGN0040: Dynamics and Vibrations
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Topics for todays class

Describing motion of particles: Motion along a curved path

Main new concept: using normal-tangential and polar coordinates

1. Review of some aspects of vectors

2. Circular Motion
   - Cartesian Coordinates
   - Normal-Tangential Coordinates

3. Motion along an arbitrary planar path: normal/tangential coordinates

4. Motion along an arbitrary planar path: polar coordinates

Quick Review of Vectors

(You can skip this part if you already know it!)

Review of some vector operations

Dot Product

Definition: $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

Cartesian component form $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ $\quad \mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$

$$\Rightarrow \mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z$$

Useful results:

- Magnitude $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{a_x^2 + a_y^2 + a_z^2}$
- Unit vector $\mathbf{n}$ parallel to a vector $\mathbf{a}$ $\quad \mathbf{n} = \mathbf{a} / |\mathbf{a}|$
- Dot products of basis vectors $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1 \quad \mathbf{i} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = \mathbf{j} \cdot \mathbf{k} = 0$

Cross Product

If $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ then

- $|\mathbf{c}| = |\mathbf{a}||\mathbf{b}|\sin\theta$
- $\mathbf{c}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$ with right hand screw convention
- Cross products of basis vectors

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$$

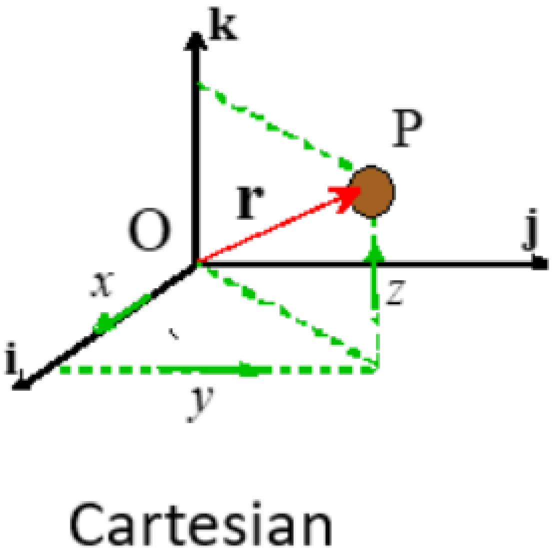
$$\mathbf{i} \times \mathbf{j} = -\mathbf{j} \times \mathbf{i} = \mathbf{k} \quad \mathbf{k} \times \mathbf{i} = -\mathbf{i} \times \mathbf{k} = \mathbf{j} \quad \mathbf{j} \times \mathbf{k} = -\mathbf{k} \times \mathbf{j} = \mathbf{i}$$

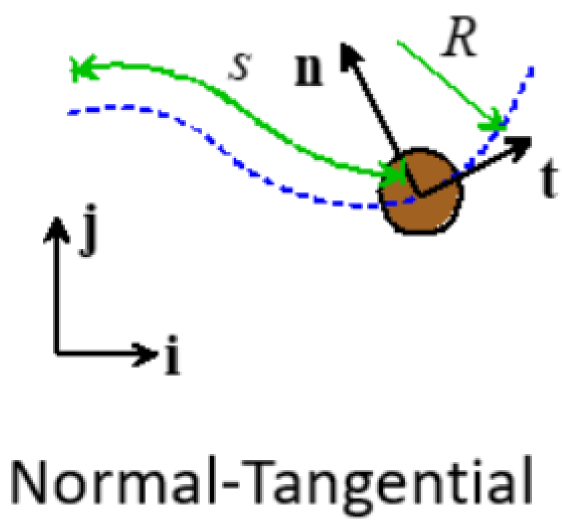
Cartesian components:

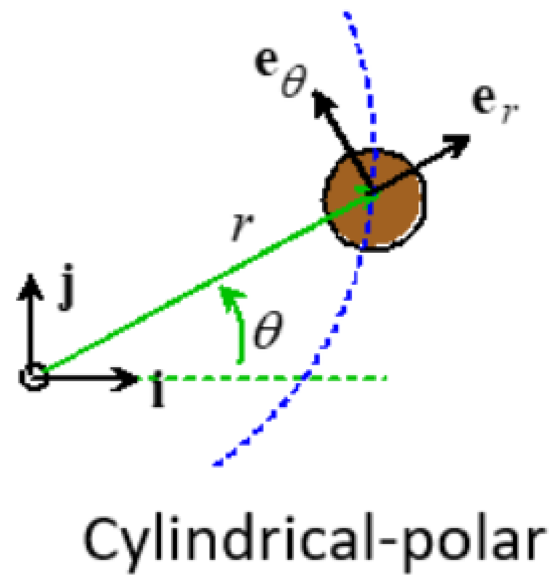
$$\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k}$$

Basis Vectors - Background

We use many different coordinate systems in dynamics:





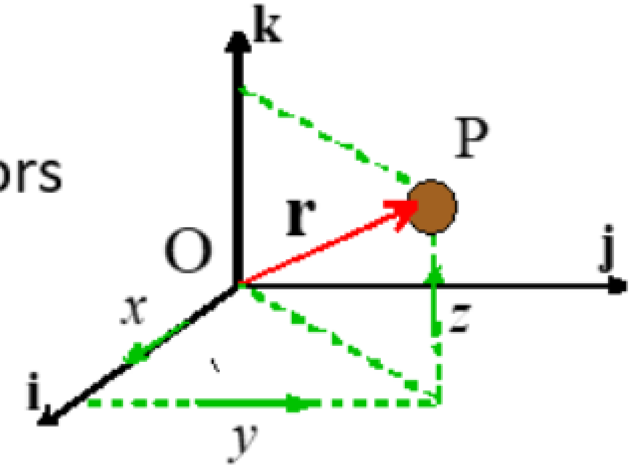


We need to understand these concepts:

1. Basis vectors
2. Components of a vector in a basis
3. How to transform components from one basis to another

Basis Vectors

Definition of a vector 'basis':



Any 3 (or 2 in 2D) linearly independent vectors
Usually (and *always* in ENGN40!):

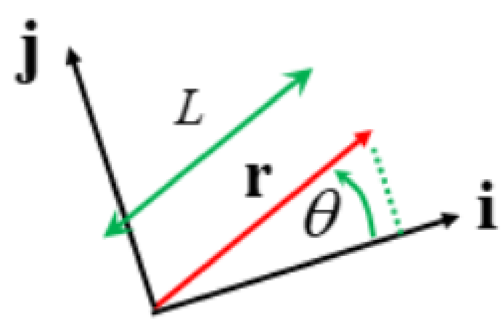
- Basis vectors have unit length
- Basis vectors are mutually perpendicular
- Example: $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ basis vectors for Cartesian components

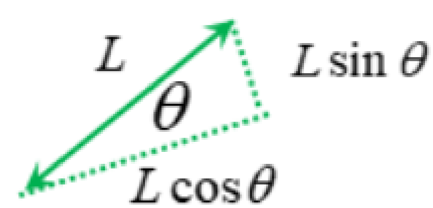
Vector components (in a basis)

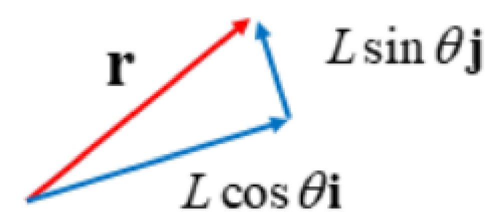
Any vector can be created by adding multiples of the basis vectors

Example: position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

We often do this by projection:







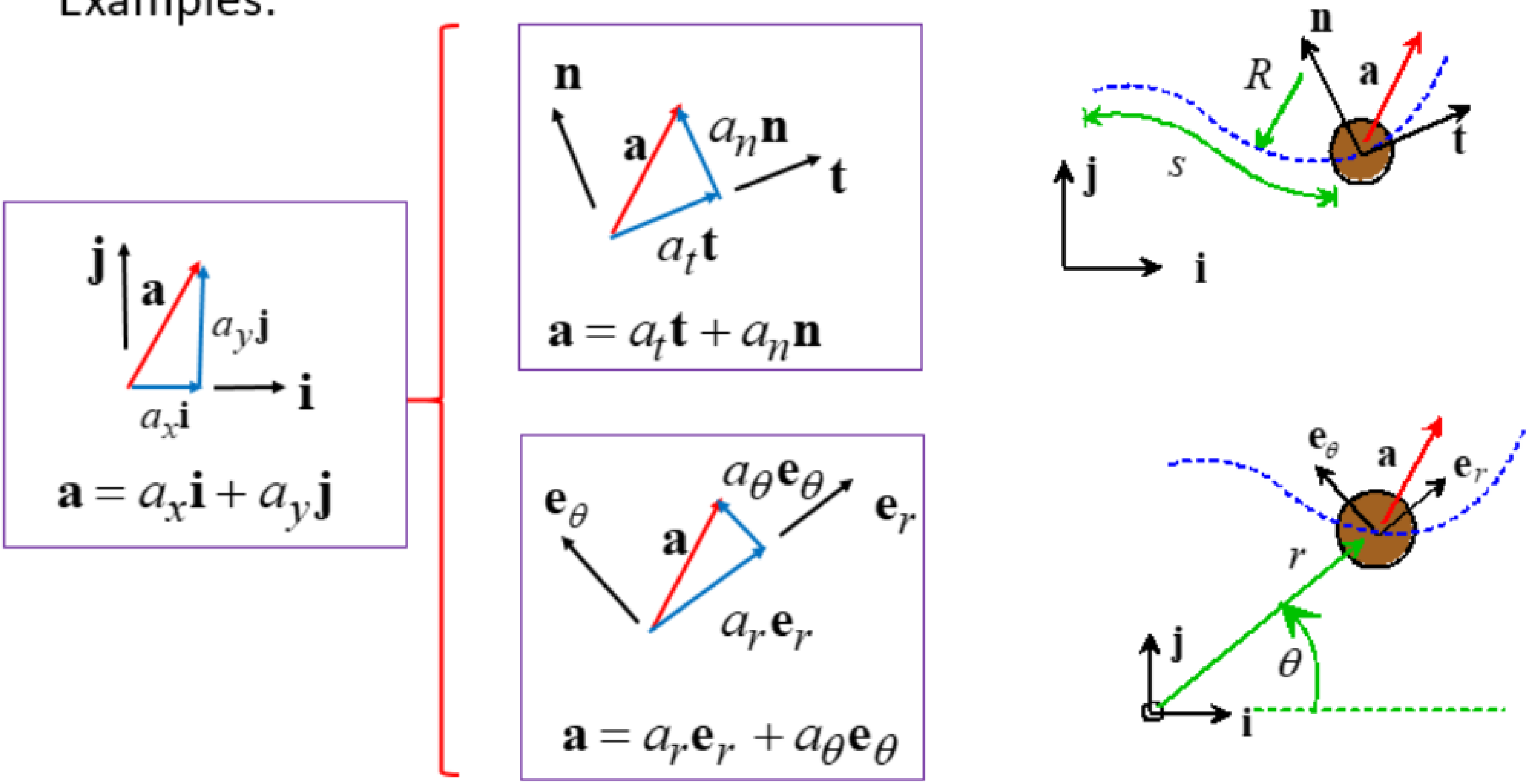
$$\mathbf{r} = L\cos\theta\,\mathbf{i} + L\sin\theta\,\mathbf{j}$$

Basis transformations

Using more than one basis:

We can express the same vector as components in more than one basis

Examples:



Basis transformations

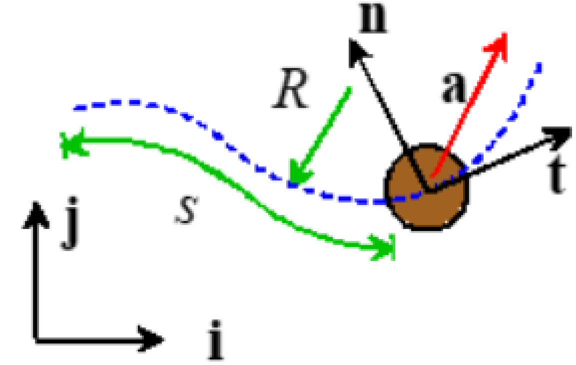
Converting from one basis to another:

Use any trick you can find!

You can often do the projection directly (use trig)

For a formal approach, use this:

To convert $\mathbf{a}$ from $\{\mathbf{i},\mathbf{j}\}$ to $\{\mathbf{n},\mathbf{t}\}$



Step 1: Write $\{\mathbf{n},\mathbf{t}\}$ in $\{\mathbf{i},\mathbf{j}\}$ components

$$\mathbf{n} = n_x\mathbf{i} + n_y\mathbf{j} \qquad \mathbf{t} = t_x\mathbf{i} + t_y\mathbf{j}$$

Step 2: Then $a_n = \mathbf{n}\cdot\mathbf{a} = n_x a_x + n_y a_y \quad a_t = \mathbf{t}\cdot\mathbf{a} = t_x a_x + t_y a_y$

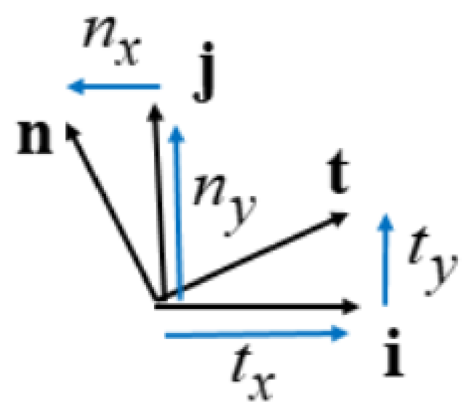
Proof: $\mathbf{a} = a_n\mathbf{n} + a_t\mathbf{t} = a_x\mathbf{i} + a_y\mathbf{j}$

$$\Rightarrow \mathbf{n}\cdot\mathbf{a} = a_n\mathbf{n}\cdot\mathbf{n} + a_t\mathbf{n}\cdot\mathbf{t} = (n_x\mathbf{i} + n_y\mathbf{j})\cdot(a_x\mathbf{i} + a_y\mathbf{j})$$

$$\Rightarrow a_n = n_x a_x + n_y a_y$$

$$\mathbf{t}\cdot\mathbf{a} = a_n\mathbf{t}\cdot\mathbf{n} + a_t\mathbf{t}\cdot\mathbf{t} = (t_x\mathbf{i} + t_y\mathbf{j})\cdot(a_x\mathbf{i} + a_y\mathbf{j})$$

$$\Rightarrow a_t = t_x a_x + t_y a_y$$



Vector operations in other bases

Use all the usual formulas for magnitude, dot and cross products.

$$\mathbf{a} = a_t\mathbf{t} + a_n\mathbf{n} + a_z\mathbf{k} \qquad \mathbf{b} = b_t\mathbf{t} + b_n\mathbf{n} + b_z\mathbf{k}$$

$$\Rightarrow \mathbf{a}\cdot\mathbf{b} = a_t b_t + a_n b_n + a_z b_z$$

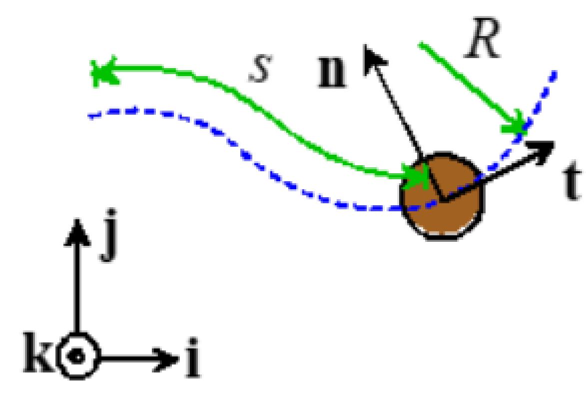
$$|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{a_t^2 + a_n^2 + a_z^2}$$

$$\mathbf{a}\times\mathbf{b} = \pm\left[(a_n b_z - a_z b_n)\mathbf{t} + (a_z b_t - a_t b_z)\mathbf{n} + (a_t b_n - a_n b_t)\mathbf{k}\right]$$

$$\mathbf{t}\times\mathbf{t} = \mathbf{n}\times\mathbf{n} = \mathbf{k}\times\mathbf{k} = 0$$

$$\mathbf{t}\times\mathbf{n} = -\mathbf{n}\times\mathbf{t} = \pm\mathbf{k} \quad \mathbf{k}\times\mathbf{t} = -\mathbf{t}\times\mathbf{k} = \pm\mathbf{n} \quad \mathbf{n}\times\mathbf{k} = -\mathbf{k}\times\mathbf{n} = \pm\mathbf{t}$$

Use + if **n** points to left of **t**, use – if **n** points to right of **t**



Normal-Tangential

$$\mathbf{a} = a_r\mathbf{e}_r + a_\theta\mathbf{e}_\theta + a_z\mathbf{k} \qquad \mathbf{b} = b_r\mathbf{e}_r + b_\theta\mathbf{e}_\theta + b_z\mathbf{k}$$

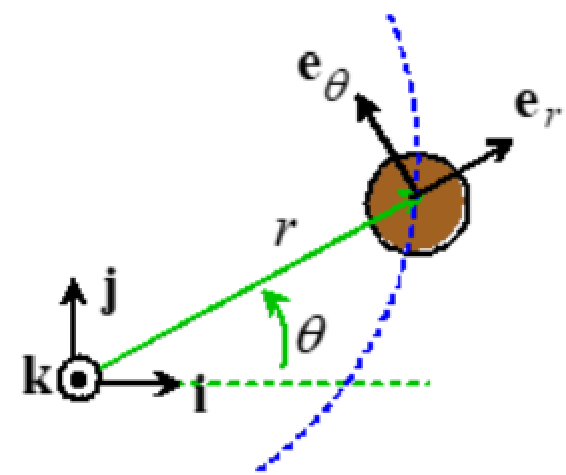
$$\Rightarrow \mathbf{a}\cdot\mathbf{b} = a_r b_r + a_\theta b_\theta + a_z b_z$$

$$|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{a_r^2 + a_\theta^2 + a_z^2}$$

$$\mathbf{a}\times\mathbf{b} = (a_\theta b_z - a_z b_\theta)\mathbf{e}_r + (a_z b_r - a_r b_z)\mathbf{e}_\theta + (a_r b_\theta - a_\theta b_r)\mathbf{k}$$

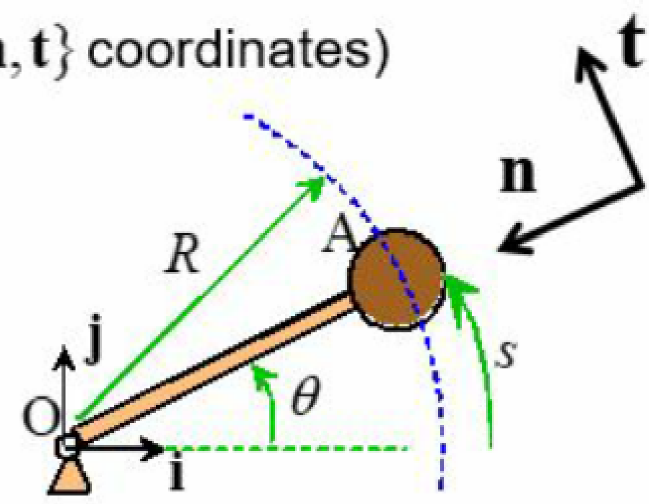
$$\mathbf{e}_r\times\mathbf{e}_r = \mathbf{e}_\theta\times\mathbf{e}_\theta = \mathbf{k}\times\mathbf{k} = 0$$

$$\mathbf{e}_r\times\mathbf{e}_\theta = -\mathbf{e}_\theta\times\mathbf{e}_r = \mathbf{k} \quad \mathbf{k}\times\mathbf{e}_r = -\mathbf{e}_r\times\mathbf{k} = \mathbf{e}_\theta \quad \mathbf{e}_\theta\times\mathbf{k} = -\mathbf{k}\times\mathbf{e}_\theta = \mathbf{e}_r$$



Cylindrical-polar

2.9: Circular motion at constant speed ($\{\mathbf{i},\mathbf{j}\}$ and $\{\mathbf{n},\mathbf{t}\}$ coordinates)



The bar OA rotates with constant angular speed ω
Find the speed, velocity and acceleration vectors of A
(give vectors in both $\{\mathbf{i},\mathbf{j}\}$ and $\{\mathbf{n},\mathbf{t}\}$ bases)

$\underline{t}$: Tangent to path

$\underline{n}$: Normal to path, towards center of circle

$\{\underline{n},\underline{t}\}$ both unit vectors

Preliminaries $\quad \theta = \omega t \qquad \omega = \text{constant}$

Geometry $\quad s = R\theta \Rightarrow s = R\omega t$

Speed $\quad V = \dfrac{ds}{dt} = R\dfrac{d\theta}{dt} \Rightarrow V = R\omega$

Position $\quad \underline{r} = R\cos\omega t\, \underline{i} + R\sin\omega t\, \underline{j}$

Velocity / Accel in Cartesian coords

Given : $\underline{r} = R\cos\omega t\,\underline{i} + R\sin\omega t\,\underline{j}$

Velocity $\quad \underline{v} = \dfrac{d\underline{r}}{dt} = -R\omega\sin\omega t\,\underline{i} + R\omega\cos\omega t\,\underline{j}$

$\Rightarrow \quad \underline{v} = R\omega(-\sin\omega t\,\underline{i} + \cos\omega t\,\underline{j})$

Recall $\quad \bar{V} = R\omega$

$$\Rightarrow \quad \underline{v} = \underbrace{\bar{V}}_{\text{magnitude}}\underbrace{(-\sin\omega t\,\underline{i} + \cos\omega t\,\underline{j})}_{\text{Direction}}$$

Acceleration

$$\underline{a} = \frac{d\underline{v}}{dt} = R\omega\left(-\omega\cos\omega t\,\underline{i} - \omega\sin\omega t\,\underline{j}\right)$$

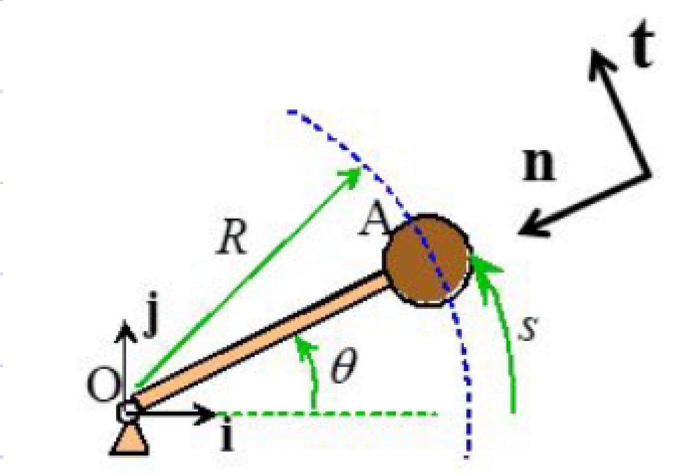
$$\Rightarrow \underline{a} = -R\omega^2\left(\cos\omega t\,\underline{i} + \sin\omega t\,\underline{j}\right)$$

Recall $\bar{V} = R\omega \Rightarrow R\omega^2 = \bar{V}\omega = \frac{V^2}{R}$

$$\Rightarrow \underline{a} = -\bar{V}\omega\left(\cos\omega t\,\underline{i} + \sin\omega t\,\underline{j}\right)$$

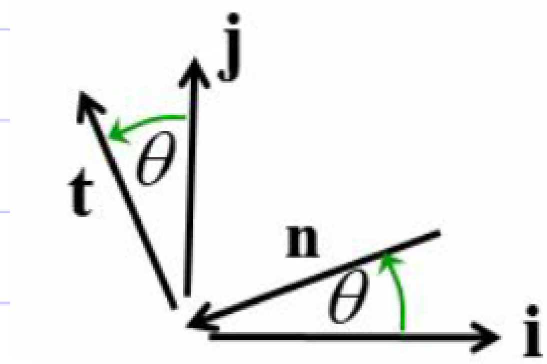
$$\underline{a} = -\frac{V^2}{R}\left(\cos\omega t\,\underline{i} + \sin\omega t\,\underline{j}\right)$$

Velocity / Accel in $\{\underline{n}, \underline{t}\}$ coords



Express $\{\underline{n}, \underline{t}\}$ in $\{\underline{i}, \underline{j}\}$ basis

Note $\underline{n}$, $\underline{t}$ have unit length



$\underline{t} = -\sin\theta\,\underline{i} + \cos\theta\,\underline{j}$ $\quad \underline{n} = -\cos\theta\,\underline{i} - \sin\theta\,\underline{j}$ $\quad \theta = \omega t$

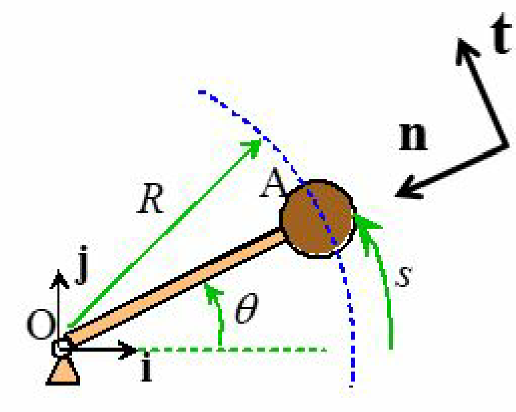
Substitute in $\{\underline{i}, \underline{j}\}$ component formulas

$\boxed{\underline{v} = R\omega\,\underline{t}}$ $\quad \boxed{\underline{v} = V\underline{t}}$

$\boxed{\underline{a} = R\omega^2\,\underline{n}}$ $\quad \boxed{\underline{a} = V\omega\,\underline{n}}$ $\quad \boxed{\underline{a} = (V^2/R)\,\underline{n}}$

NB: V is constant in these formulas

Interpreting the velocity / accel formulas



$\underline{V} = V\,\underline{t}$

Magnitude of velocity = speed
Direction is tangent to path

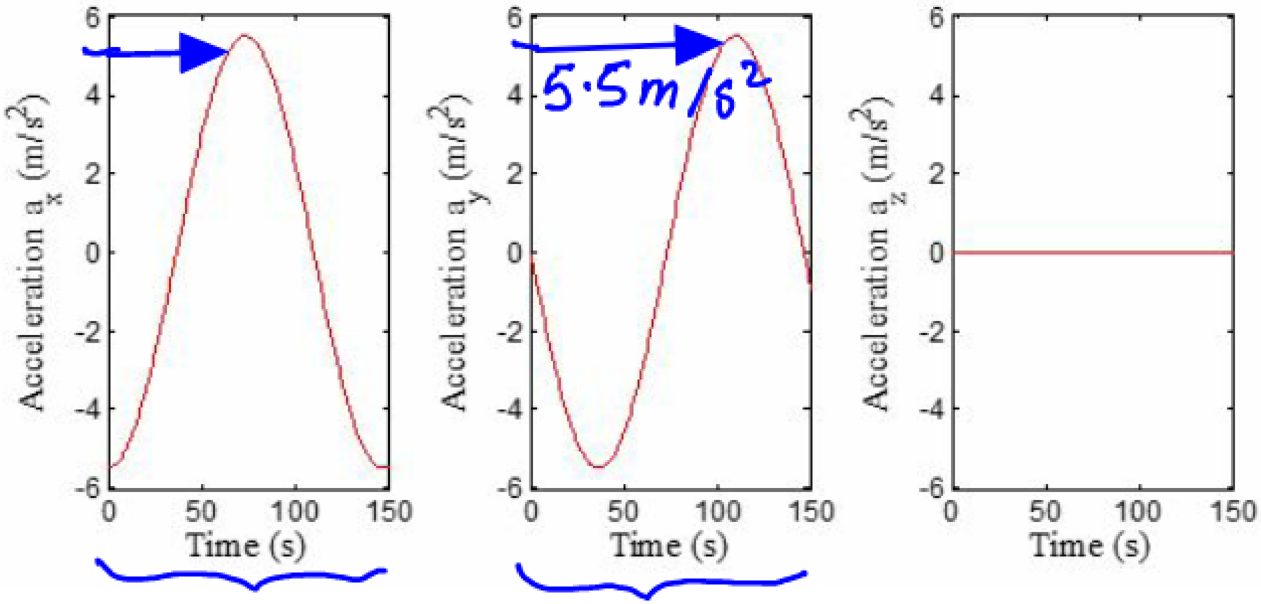
$$\underline{a} = \frac{V^2}{R}\,\underline{n}$$

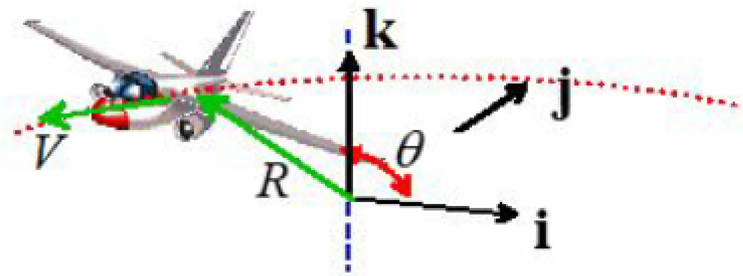
Magnitude of acceleration is V^2/R
Direction is normal to path, towards center of circle

Note $\underline{a} \neq \underline{0}$ since direction of $\underline{V}$ changes

2.10 Example: Interpreting acceleration data from an inertial platform

An inertial platform (with fixed orientation) records the accelerations shown. Determine: (a) The radius of the path; and (b) The aircrafts speed





1 cycle in 150 s

Measurement: $\underline{a} = -5.5 \cos \frac{2\pi t}{150} \underline{i} - 5.5 \sin \frac{2\pi t}{150} \underline{j}$

Formula $\underline{a} = -R\omega^2 \cos \omega t \, \underline{i} - R\omega^2 \sin \omega t \, \underline{j}$

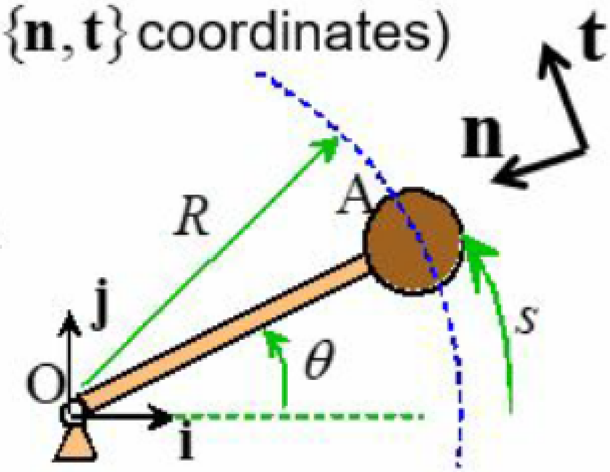
Compare: $R\omega^2 = 5.5 \quad \omega = 2\pi/150 \Rightarrow R = 5.5\left(\frac{150}{2\pi}\right)^2$

$\Rightarrow \boxed{R = 3.1 \text{ km}} \quad \boxed{V = R\omega = 129 \text{ m/s}}$

2.11: Circular motion at arbitrary speed ($\{\mathbf{i},\mathbf{j}\}$ and $\{\mathbf{n},\mathbf{t}\}$ coordinates)

The angle $\theta(t)$ is an arbitrary function of time
Find the speed, velocity and acceleration vectors of A
(give vectors in both $\{\mathbf{i},\mathbf{j}\}$ and $\{\mathbf{n},\mathbf{t}\}$ bases)

$$\mathbf{t} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j} \qquad \mathbf{n} = -\cos\theta\mathbf{i} - \sin\theta\mathbf{j}$$



Definitions $\quad \omega = \dfrac{d\theta}{dt} \quad \alpha = \dfrac{d\omega}{dt}$ "angular accel"

Geometry: $s = R\theta$

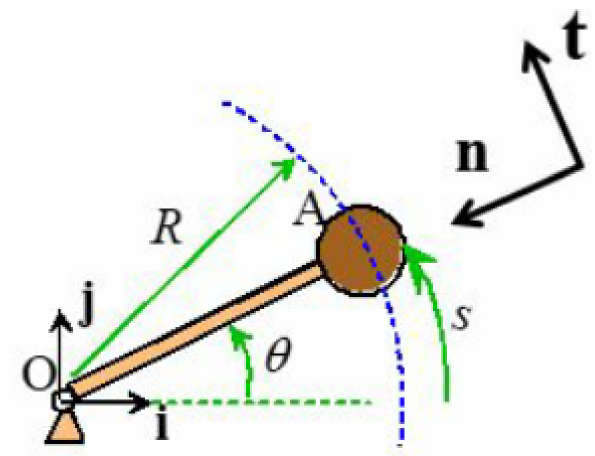
Speed $\quad V = \dfrac{ds}{dt} = R\dfrac{d\theta}{dt} \Rightarrow \boxed{V = R\omega}$

Define: "Tangential accel" $\boxed{a_t = \dfrac{dV}{dt} = R\dfrac{d\omega}{dt} = R\alpha}$

Also useful $\boxed{a_t = \dfrac{dV}{ds}\dfrac{ds}{dt} = V\dfrac{dV}{ds}}$

Position

$$\underline{r} = R\cos\theta\,\underline{i} + R\sin\theta\,\underline{j}$$



$$\mathbf{t} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$$

$$\mathbf{n} = -\cos\theta\mathbf{i} - \sin\theta\mathbf{j}$$

Velocity

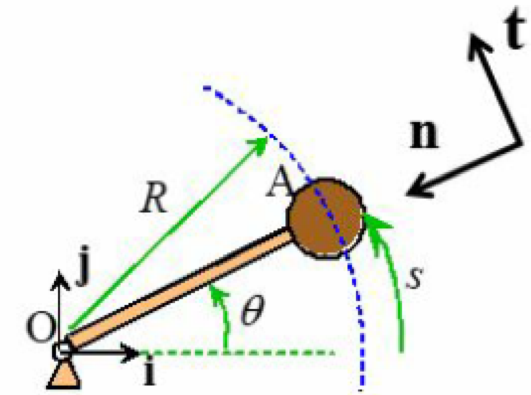
$$\underline{V} = \frac{d\underline{r}}{dt} = -R\frac{d\theta}{dt}\sin\theta\,\underline{i} + R\frac{d\theta}{dt}\cos\theta\,\underline{j}$$

$$\Rightarrow \quad \underline{V} = R\omega\,(-\sin\theta\,\underline{i} + \cos\theta\,\underline{j})$$

$$\Rightarrow \quad \underline{V} = R\omega\,\underline{t}$$

$$\Rightarrow \quad \underline{V} = V\,\underline{t}$$

Acceleration



$\mathbf{t} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$

$\mathbf{n} = -\cos\theta\,\mathbf{i} - \sin\theta\,\mathbf{j}$

$$\underline{a} = \frac{d\underline{V}}{dt}$$

$$= \frac{d}{dt}\left\{ R\omega\left(-\sin\theta\,\underline{i} + \cos\theta\,\underline{j}\right)\right\}$$

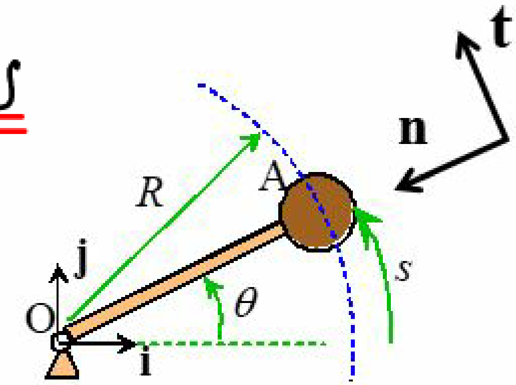
$$= R\frac{d\omega}{dt}\left(-\sin\theta\,\underline{i} + \cos\theta\,\underline{j}\right) + R\omega\left(-\frac{d\theta}{dt}\cos\theta\,\underline{i} - \frac{d\theta}{dt}\sin\theta\,\underline{j}\right)$$

$$\Rightarrow \underline{a} = R\alpha\left(-\sin\theta\,\underline{i} + \cos\theta\,\underline{j}\right) - R\omega^2\left(\cos\theta\,\underline{i} + \sin\theta\,\underline{j}\right)$$

$$\Rightarrow \underline{a} = R\alpha\,\underline{t} + R\omega^2\,\underline{n}$$

$$\Rightarrow \underline{a} = \frac{d\bar{V}}{dt}\,\underline{t} + \frac{\bar{V}^2}{R}\,\underline{n}$$

Interpreting the acceleration formulas



$\mathbf{t} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$

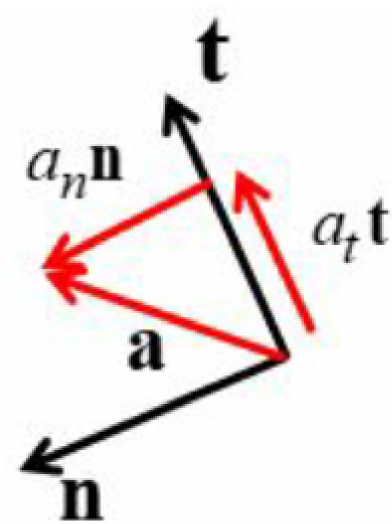
$\mathbf{n} = -\cos\theta\mathbf{i} - \sin\theta\mathbf{j}$

Define

$a_t = \dfrac{d\bar{V}}{dt}$ "tangential accel"

$a_n = \dfrac{V^2}{R}$ "normal accel"

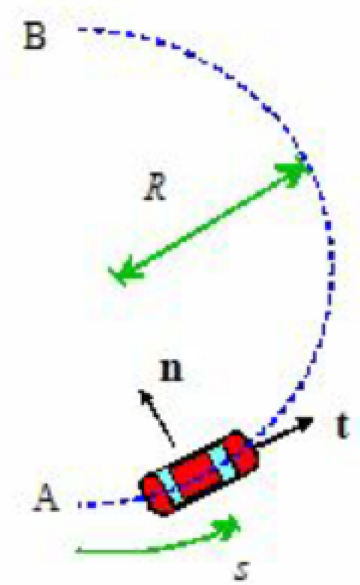
Then $\underline{a} = \underbrace{a_t \underline{t}}_{\text{Component parallel to motion}} + \underbrace{a_n \underline{n}}_{\text{Component perpendicular to motion}}$



a_t : Represents change in speed

a_n : Represents change in direction

2.12: Example: Vehicle accelerating around a curve



The vehicle starts at rest at A and travels with constant tangential acceleration a_t

Find a formula for the magnitude of the acceleration at B, in terms of a_t

Formula : $\underset{\sim}{a} = a_t \underset{\sim}{t} + \dfrac{V^2}{R}\underset{\sim}{n} \Rightarrow |\underset{\sim}{a}| = \sqrt{a_t^2 + \left(\dfrac{V^2}{R}\right)^2}$

Need to find V.

Recall $a_t = V\dfrac{dV}{ds}$ At B $s = \pi R$

Separate variables

$$\int_0^{V_B} V\,dV = \int_0^{\pi R} a_t\,ds \Rightarrow \frac{1}{2}V_B^2 = \pi R a_t$$

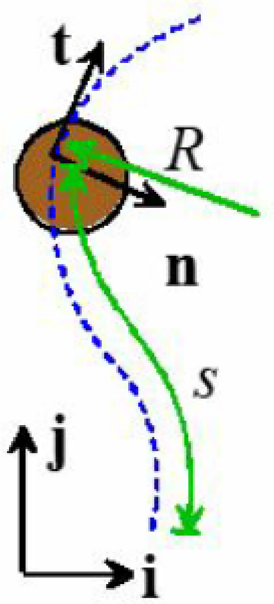
Hence $\vec{V}_B = \sqrt{2\pi R a_t}$

Substitute into formula for $|\underline{a}|$

$$|\underline{a}| = \sqrt{a_t^2 + \left(\frac{\vec{V}_B^2}{R}\right)^2} = \sqrt{a_t^2 + \frac{(2\pi R a_t)^2}{R^2}}$$

$$\Rightarrow \boxed{|\underline{a}| = a_t\sqrt{1 + 4\pi^2}}$$

2.13 Motion along an arbitrary planar path using $\{\underline{n}, \underline{t}\}$ coordinates



Distance traveled $s(t)$

Path $\mathbf{r} = x(s)\mathbf{i} + y(s)\mathbf{j}$

Speed $V = \dfrac{ds}{dt}$

Radius of curvature

$$R = \frac{1}{\sqrt{\left(\dfrac{d^2x}{ds^2}\right)^2 + \left(\dfrac{d^2y}{ds^2}\right)^2}}$$

$\underline{t}$: Tangent to path

$\underline{n}$: Normal to path, towards center of curvature

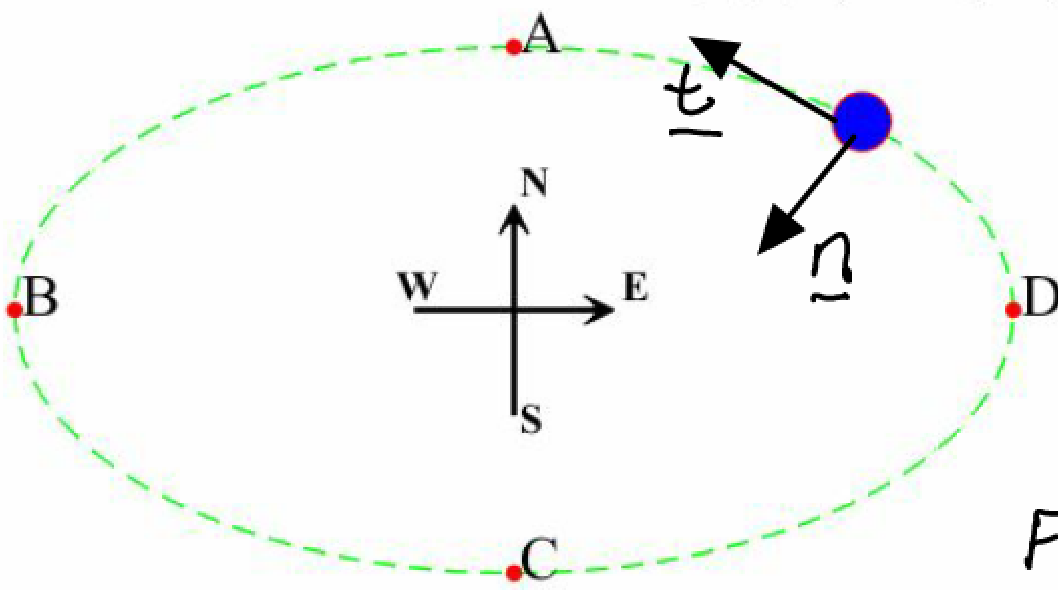
Every point on path is locally a circle

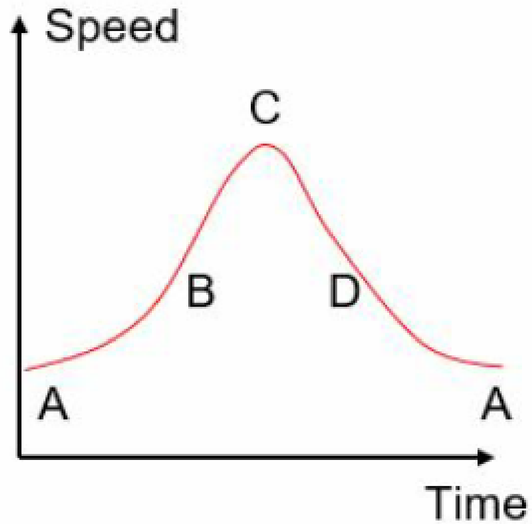
Use circular motion formulas with R = radius of curvature

$$\boxed{\underline{v} = V\underline{t}} \qquad \boxed{\underline{a} = \frac{dV}{dt}\underline{t} + \frac{V^2}{R}\underline{n}}$$

Concept question

Give the direction of the acceleration at A,B,C,D (on compass)



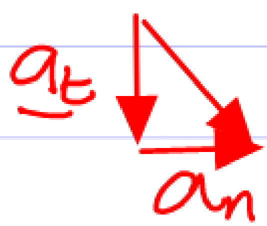


Formula $\underline{a} = \frac{dV}{dt}\underline{t} + \frac{V^2}{R}\underline{n}$

At A: V is min $\Rightarrow dV/dt = 0$

$\Rightarrow$ $\underline{a}$ is South

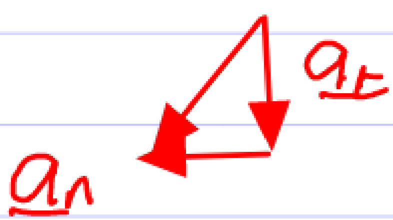
At B $dV/dt > 0$



$\Rightarrow$ $\underline{a}$ is SE

At C $dV/dt = 0$ $\Rightarrow$ $\underline{a}$ is North

At D $dV/dt < 0$



$\Rightarrow$ $\underline{a}$ is SW

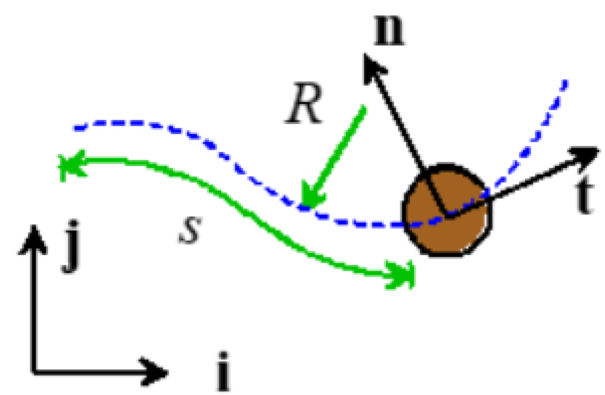
Deriving the n-t coordinate formulas

Formula for the path (assumed to be given):

Distance travelled $s(t)$, $\quad \mathbf{r} = x(s)\mathbf{i} + y(s)\mathbf{j}$

Radius of curvature

$$R = \frac{1}{\sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2}}$$



Speed $\quad V = \dfrac{ds}{dt}$

Definitions of $\{\mathbf{n}, \mathbf{t}\}$ $\quad \mathbf{t} = \dfrac{d\mathbf{r}}{ds} \qquad \mathbf{n} = R\dfrac{d\mathbf{t}}{ds} = R\dfrac{d^2\mathbf{r}}{ds^2}$

Velocity $\quad \mathbf{v} = \dfrac{d\mathbf{r}}{dt} = \dfrac{d\mathbf{r}}{ds}\dfrac{ds}{dt} \Rightarrow \mathbf{v} = V\mathbf{t}$

Acceleration

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d(V\mathbf{t})}{dt} = \frac{dV}{dt}\mathbf{t} + V\frac{d\mathbf{t}}{dt} = \frac{dV}{dt}\mathbf{t} + V\frac{d\mathbf{t}}{ds}\frac{ds}{dt}$$

$$\Rightarrow \mathbf{a} = \frac{dV}{dt}\mathbf{t} + \frac{V^2}{R}\mathbf{n}$$

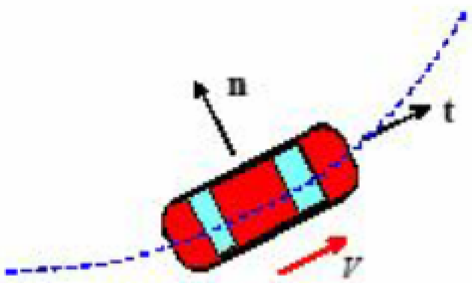
Radius of curvature $\quad \mathbf{n}\cdot\mathbf{n} = 1 \Rightarrow R^2\dfrac{d^2\mathbf{r}}{ds^2}\cdot\dfrac{d^2\mathbf{r}}{ds^2} = 1$

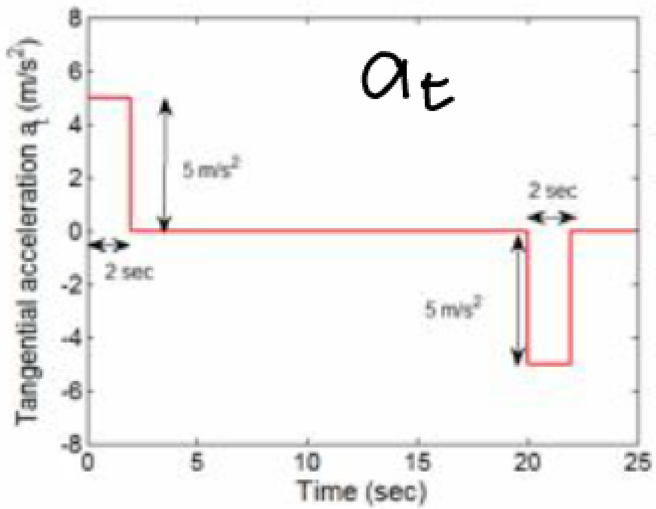
Note: $\quad \dfrac{d^2\mathbf{r}}{ds^2} = \dfrac{d^2x}{ds^2}\mathbf{i} + \dfrac{d^2y}{ds^2}\mathbf{j} \qquad \Rightarrow \qquad \dfrac{1}{R^2} = \left(\dfrac{d^2x}{ds^2}\right)^2 + \left(\dfrac{d^2y}{ds^2}\right)^2$

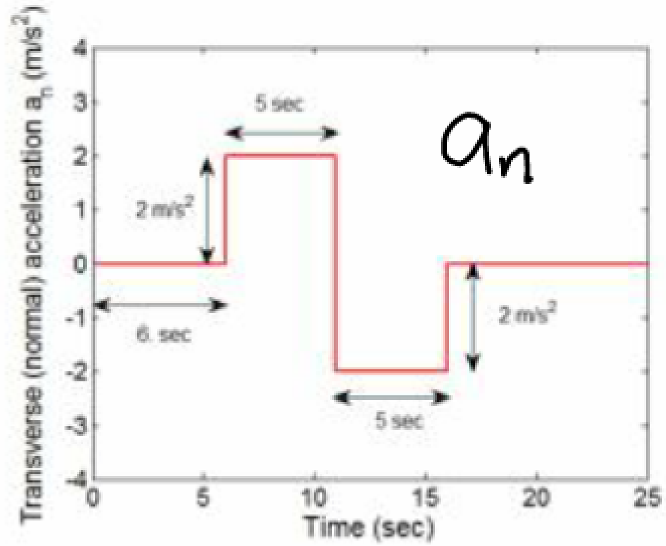
2.13: Example: Reconstructing a path from acceleration data

A vehicle is instrumented to measure acceleration components in directions parallel and perpendicular to the car's direction of motion. (A positive transverse accel means the car accelerates to the left). The Car is at rest at the origin at time *t=0* facing the *x* direction.

1. Sketch a graph showing the car's speed as a function of time.
2. Sketch the subsequent path of the vehicle







Formulas: $a_t = \frac{dV}{dt}$

$$\underset{\sim}{a} = \underbrace{\frac{dV}{dt}}_{a_t}\,\underset{\sim}{t} + \underbrace{\frac{V^2}{R}}_{a_n}\,\underset{\sim}{n}$$

Part (1): Use graphical method

We know $\dfrac{dV}{dt} = a_t$

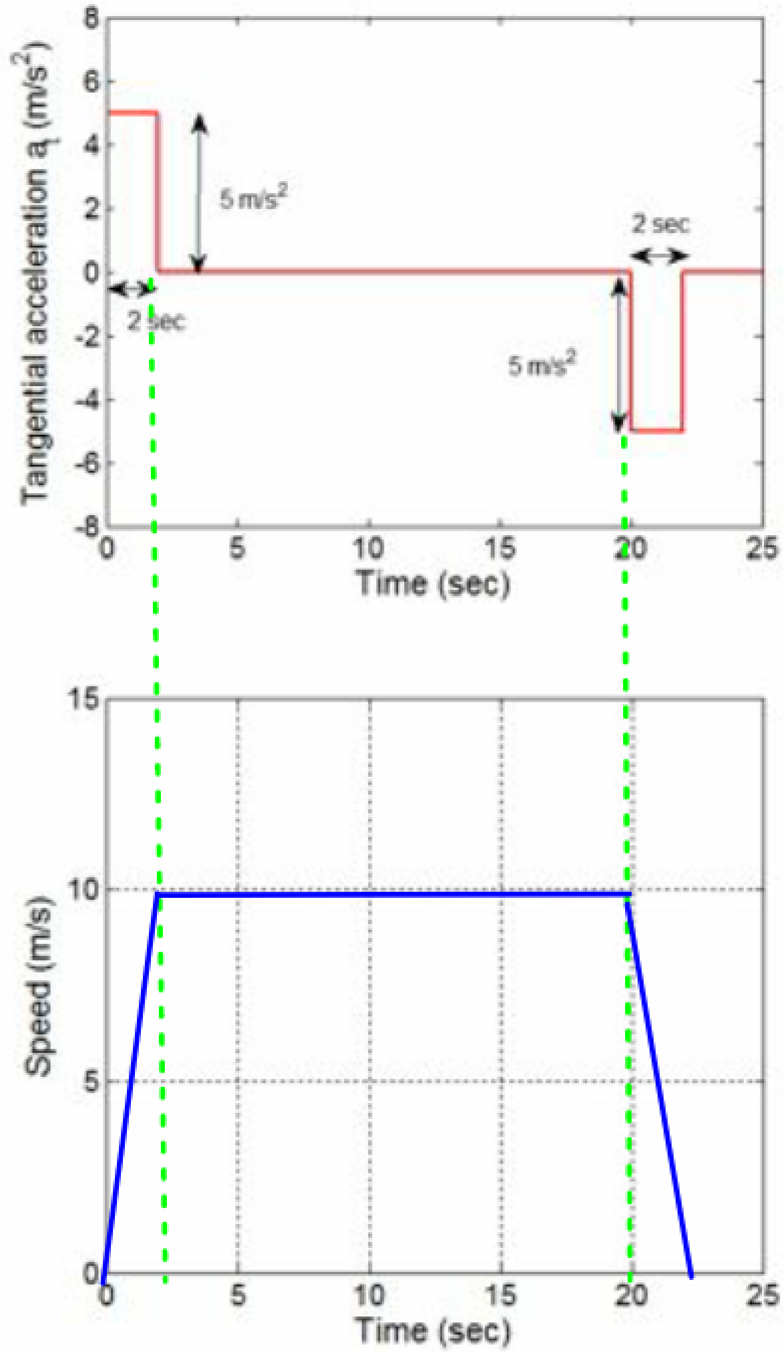
a_t given - find V by integrating

Const accel $\Rightarrow$ V is a straight line with slope $= a_t$

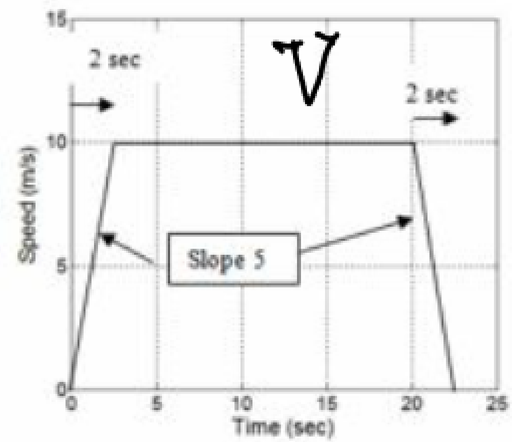
Zero $a_t \Rightarrow V$ is const

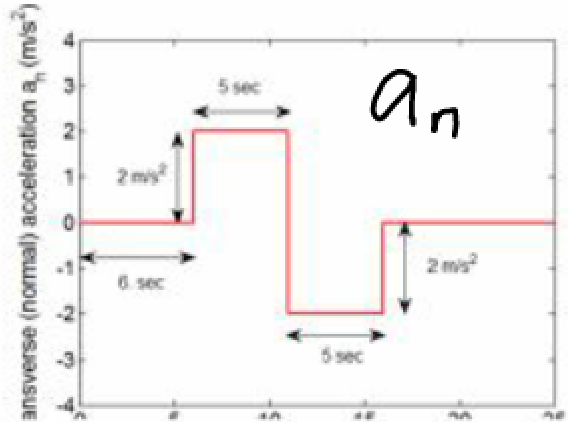
At time $t = 2s$

$V = a_t t = 5 \times 2 = 10\ m/s$



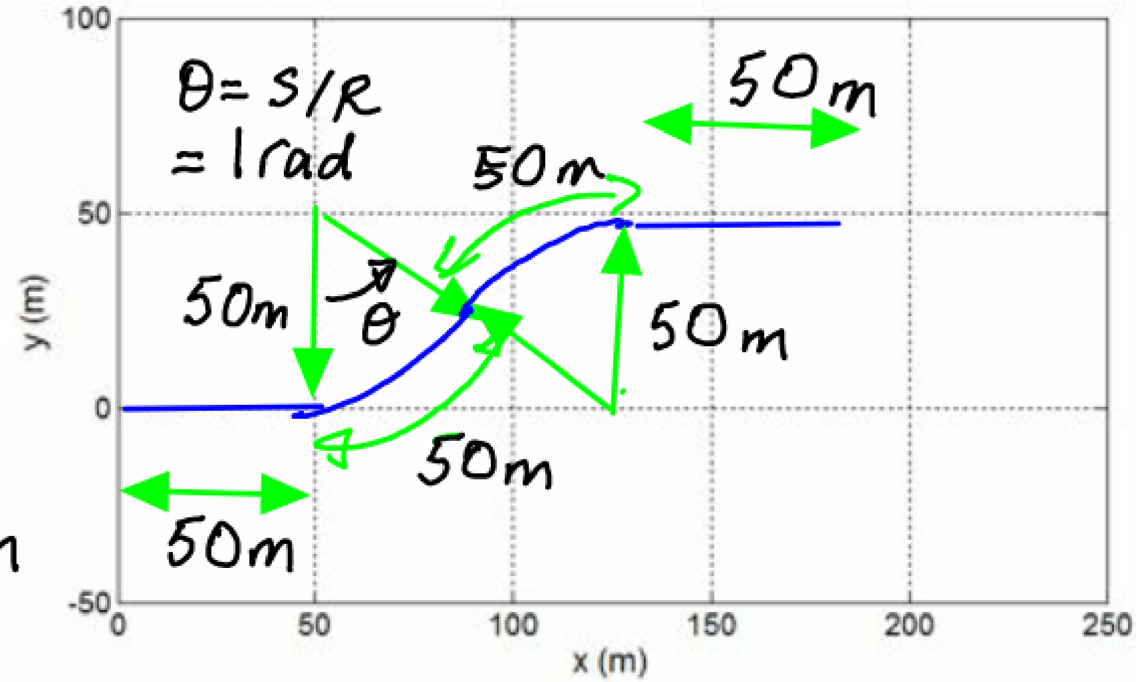
Part 2





Recall $a_n = V^2/R$

$\Rightarrow R = V^2/a_n$



$0 < t < 6s$

$a_n = 0 \Rightarrow R \to \infty$

$\Rightarrow$ straight line

Dist travelled = area under V-t graph

$= \frac{1}{2} \times 2 \times 10 + 4 \times 10 = 50m$

$6 < t < 11s$

$a_n = 2\ m/s^2 \quad V = 10\ m/s \Rightarrow R = 10^2/2 = 50m$

Dist travelled $= 10\ m/s \times 5s = 50m$

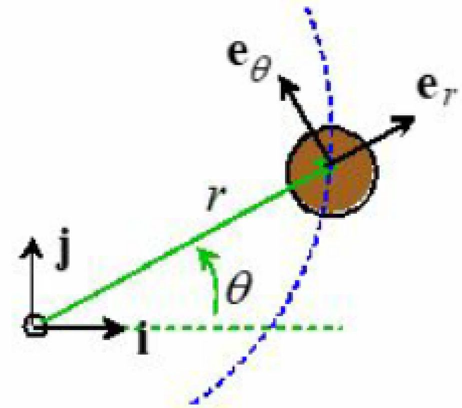
$11s < t < 25s$: path repeats, turning right

2.15: Analyzing motion using polar coords

Problem: Given $r(t)$, $\theta(t)$

Find $\underline{r}$, $\underline{v}$, $\underline{a}$

in $\{\underline{e_r}, \underline{e_\theta}, \underline{k}\}$



$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}(y/x)$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Solution

Position $\underline{r} = r(t)\,\underline{e_r}$

Velocity $\underline{v} = \frac{dr}{dt}\underline{e_r} + r\frac{d\theta}{dt}\underline{e_\theta}$

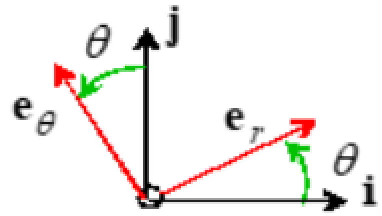
Acceleration

$$\underline{a} = \left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\}\underline{e_r} + \left\{r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right\}\underline{e_\theta}$$

Deriving the polar coordinate formulas

Time derivatives of $\{\mathbf{e}_r, \mathbf{e}_\theta\}$

$$\mathbf{e}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$$
$$\mathbf{e}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$$



$$\frac{d\mathbf{e}_r}{dt} = -\frac{d\theta}{dt}\sin\theta\mathbf{i} + \frac{d\theta}{dt}\cos\theta\mathbf{j} \quad = \frac{d\theta}{dt}\mathbf{e}_\theta$$
$$\frac{d\mathbf{e}_\theta}{dt} = -\frac{d\theta}{dt}\cos\theta\mathbf{i} - \frac{d\theta}{dt}\sin\theta\mathbf{j} \quad = -\frac{d\theta}{dt}\mathbf{e}_r$$

Position $\quad \mathbf{r} = r(t)\mathbf{e}_r(t)$

Velocity $\quad \mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\mathbf{e}_r}{dt} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta$

Acceleration

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2r}{dt^2}\mathbf{e}_r + \frac{dr}{dt}\frac{d\mathbf{e}_r}{dt} \quad + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{e}_\theta + r\frac{d^2\theta}{dt^2}\mathbf{e}_\theta + r\frac{d\theta}{dt}\frac{d\mathbf{e}_\theta}{dt}$$
$$= \frac{d^2r}{dt^2}\mathbf{e}_r + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{e}_\theta + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{e}_\theta + r\frac{d^2\theta}{dt^2}\mathbf{e}_\theta - r\left(\frac{d\theta}{dt}\right)^2\mathbf{e}_r$$
$$= \left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\}\mathbf{e}_r \quad + \quad \left\{r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right\}\mathbf{e}_\theta$$

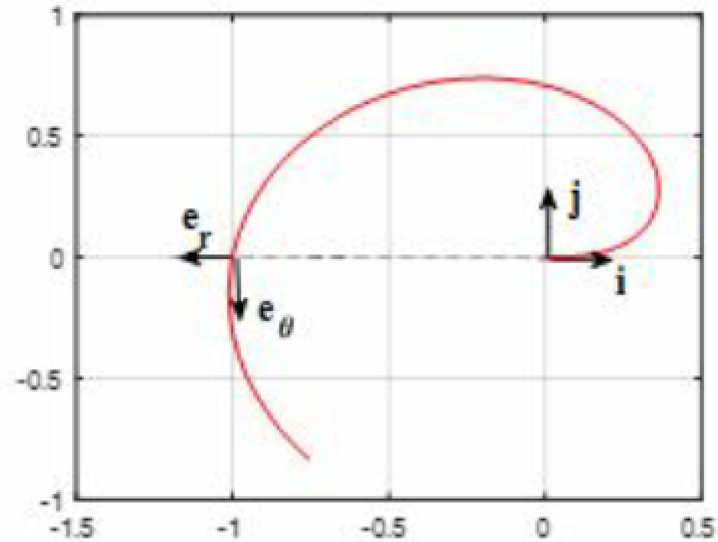
2.16 Example: The particle has polar coordinates

$$\theta = t^2 \qquad r = t/\sqrt{\pi}$$

At the instant when $\theta = \pi$ calculate

(i) the position, velocity and acceleration vectors in the polar basis $\mathbf{e}_r, \mathbf{e}_\theta$

(ii) The normal and tangential components of acceleration



Preliminaries: Note $t = \sqrt{\theta} \Rightarrow$ when $\theta = \pi$ $t = \sqrt{\pi}$

$dr/dt = 1/\sqrt{\pi} \qquad d^2r/dt^2 = 0 \qquad d\theta/dt = 2t \qquad d^2\theta/dt^2 = 2$

Position $\quad \underline{r} = r\underline{e_r} = \underline{e_r}$

Velocity $\quad \underline{v} = dr/dt\, \underline{e_r} + r\, d\theta/dt\, \underline{e_\theta}$

$$\Rightarrow \quad \underline{v} = \frac{1}{\sqrt{\pi}}\underline{e_r} + 2\sqrt{\pi}\,\underline{e_\theta}$$

Acceleration

$$\underline{a} = \left\{ \frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \right\} \underline{e_r} + \left\{ r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} \right\} \underline{e_\theta}$$

$$\underline{a} = \left(0 - 1 \times (2\sqrt{\pi})^2\right) \underline{e_r} + \left(1 \times 2 + 2 \frac{1}{\sqrt{\pi}} 2\sqrt{\pi}\right) \underline{e_\theta}$$

$$\boxed{\underline{a} = 4\pi \, \underline{e_r} + 6 \, \underline{e_\theta}}$$

$\underline{n}, \underline{t}$ components

(1) Find $\{\underline{n}, \underline{t}\}$ in $\{\underline{e_r}, \underline{e_\theta}\}$

(2) Use $a_t = \underline{t} \cdot \underline{a}$ $\quad a_n = \underline{n} \cdot \underline{a}$

Finding $\{\underline{n}, \underline{t}\}$

Recall $\underline{v} = V\underline{t} \Rightarrow \underline{t} = \underline{v} / V$

and V = speed = $|\underline{v}|$

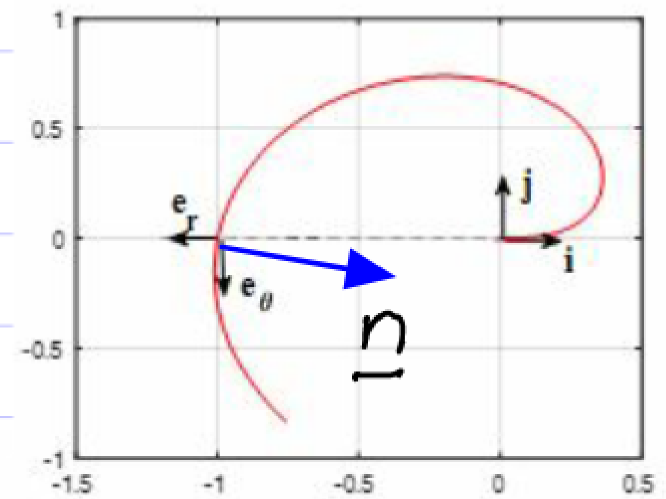
$$\Rightarrow \underline{t} = \frac{(1/\sqrt{\pi})\,\underline{e_r} + 2\sqrt{\pi}\,\underline{e_\theta}}{\sqrt{1/\pi + 4\pi}}$$

$$= (\underline{e_r} + 2\pi\,\underline{e_\theta}) \,/\, \sqrt{1 + 4\pi^2}$$

$\underline{n}$ is perpendicular to $\underline{t}$ and $\underline{k}$

$$\Rightarrow \underline{n} = \pm\, \underline{k} \times \underline{t}$$

Choose sign so $\underline{n}$ has negative $\underline{e_r}$ component



$$\Rightarrow \underline{n} = \pm \underline{k} \times \underline{t} = \pm \underline{k} \times (\underline{e_r} + 2\pi \underline{e_\theta}) / \sqrt{1+4\pi^2}$$

Note $\underline{k} \times \underline{e_r} = \underline{e_\theta}$ $\quad \underline{k} \times \underline{e_\theta} = -\underline{e_r}$

$$\Rightarrow \underline{n} = \pm (\underline{e_\theta} - 2\pi \underline{e_r}) / \sqrt{1+4\pi^2}$$

Choose + sign $\Rightarrow$ $\underline{n} = (-2\pi \underline{e_r} + \underline{e_\theta}) / \sqrt{1+4\pi^2}$

Finally $a_t = \underline{a} \cdot \underline{t} = (-4\pi \underline{e_r} + 6 \underline{e_\theta}) \cdot \dfrac{(\underline{e_r} + 2\pi \underline{e_\theta})}{\sqrt{1+4\pi^2}}$

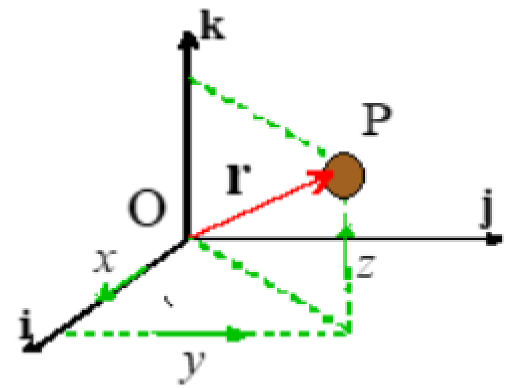
$$\Rightarrow \boxed{a_t = \frac{(-4\pi + 12\pi)}{\sqrt{1+4\pi^2}} = \frac{8\pi}{\sqrt{1+4\pi^2}}}$$

$$\boxed{a_n = \underline{a} \cdot \underline{n} = \frac{(8\pi^2 + 6)}{\sqrt{1+4\pi^2}}}$$

Newton's laws in non-inertial bases

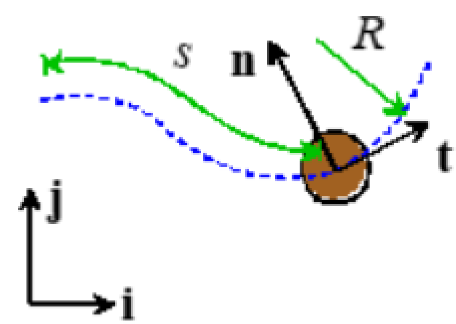
Note $\{\mathbf{n},\mathbf{t}\}$ and $\{\mathbf{e}_r,\mathbf{e}_\theta\}$ are not inertial bases

Why can we still use Newton's laws?



To use $\{\mathbf{n},\mathbf{t}\}$ or $\{\mathbf{e}_r,\mathbf{e}_\theta\}$ we always have to choose the inertial $\{\mathbf{i},\mathbf{j}\}$ basis first

We then convert Newton's laws from $\{\mathbf{i},\mathbf{j}\}$ to $\{\mathbf{n},\mathbf{t}\}$ and $\{\mathbf{e}_r,\mathbf{e}_\theta\}$

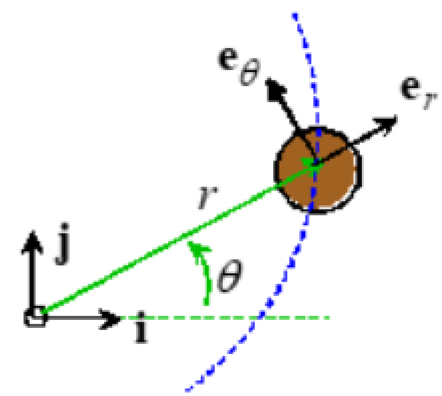


Normal-Tangential

$$F_t\mathbf{t} + F_n\mathbf{n} = m\left(\frac{dV}{dt}\mathbf{t} + \boxed{\frac{V^2}{R}\mathbf{n}}\right)$$

$$F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta = m\left\{\frac{d^2r}{dt^2} - \boxed{r\left(\frac{d\theta}{dt}\right)^2}\right\}\mathbf{e}_r + \boxed{m\left\{r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right\}\mathbf{e}_\theta}$$

Additional terms resulting from nonzero time derivatives of basis vectors



Cylindrical-polar